Quantization of a free scalar field

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Commutation relations

First quantization : $p^i, q^i \to [\hat{q}^i, \hat{p}^j] = i\delta_{ij}$

Second quantization: $\phi(t,\bar{x}),\ \pi(t,\bar{x}) \to [\hat{\phi}(t,\bar{x}), \hat{\pi}(t,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y})$

$$[\hat{\phi}(t,\bar{x}), \hat{\phi}(t,\bar{y})] = [\hat{\pi}(t,\bar{x}), \hat{\pi}(t,\bar{y})] = 0$$

$\phi(t,\bar{x})$ real field $\to$ $\hat{\phi}(t,\bar{x}) = \hat{\phi}^+(t,\bar{x})$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}}\left(a_p e^{-ipx} + a_p^+ e^{+ipx}\right) \quad \text{with } p^0 = E_p \equiv (m^2+\bar{p}^2)^{1/2}$$

– Show that $[a_p, a_q^+] = (2\pi)^3\delta^{(3)}(\bar{p}-\bar{q})$, $[a_p, a_q] = [a_p^+, a_q^+] = 0$ implies

$$[\phi(t,\bar{x}), \pi(t,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y}) \text{ and } [\phi(t,\bar{x}), \phi(t,\bar{y})] = [\pi(t,\bar{x}), \pi(t,\bar{x})] = 0$$

for $\pi(t,\bar{x}) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \dot{\phi}(t,\bar{x})$

$$\pi(t,\bar{x}') = \int \frac{d^3p'}{(2\pi)^3(2E_{p'})^{1/2}}(-iE_{p'})\left(a_{p'}e^{-ip'x'} - a_{p'}^+e^{ip'x'}\right) \qquad x' \equiv (t,\bar{x}')$$

$$[\phi(t,\bar{x}), \pi(t,\bar{x}')] = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}}\frac{d^3p'}{(2\pi)^3(2E_{p'})^{1/2}}(-iE_{p'})\left[a_p e^{-ipx} + a_p^+e^{ipx}, a_{p'}e^{-ip'x'} - a_{p'}^+e^{ip'x'}\right] =$$

$$= \text{—} \ \text{''} \text{——} \text{''} \text{——} \text{''} \text{—} \left\{-\underbrace{[a_p, a_{p'}^+]}_{(2\pi)^3\delta^{(3)}(\bar{p}-\bar{p}')} e^{-ipx+ip'x'} + \underbrace{[a_p^+, a_{p'}]}_{-(2\pi)^3\delta^{(3)}(\bar{p}'-\bar{p})} e^{ipx-ip'x'}\right\} =$$

$$= \int \frac{d^3p}{(2\pi)^3(2\cancel{E_p})^{1/2}}\frac{-i\cancel{E_p}}{(2\cancel{E_p})^{1/2}}(-)\left\{e^{i\bar{p}(\bar{x}-\bar{x}')} + e^{-\bar{p}(\bar{x}-\bar{x}')}\right\} = i\frac{1}{\cancel{2}}\cancel{2}\delta^{(3)}(\bar{x}-\bar{x}')$$

$$\delta^{(3)}(\bar{p}-\bar{p}') \Rightarrow -i(px-p'x') = +i\bar{p}(\bar{x}-\bar{x}')$$

$$E_p = E_{p'}$$

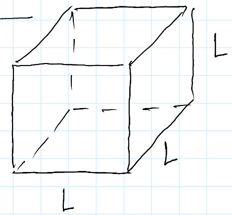
$$\int dk\, e^{ikx} = 2\pi\delta(x)$$

$E_p = E_{p'}$

$$\int dk\, e^{ikx} = 2\pi\,\delta(x)$$

Box normalization

$\mathbb{R}^3 \longrightarrow$ box



with periodic boundary conditions $\phi(x^i) = \phi(x^i + L)$

$$e^{ip^i x^i} = e^{ip^i(x^i+L)}$$
$$\Downarrow$$
$$p^i L = 2\pi n_i \qquad n_i = 0, \pm 1, \dots$$

$$\bar{p}_{\bar{n}} = \frac{2\pi}{L}\underbrace{(n_1, n_2, n_3)}_{\overset{\shortparallel}{\bar{n}}}$$
$$\Downarrow$$
$$\int d^3p \rightarrow \left(\frac{2\pi}{L}\right)^3 \sum_{\bar{n}}$$

$$\int d^3p\, \delta^{(3)}(\bar{p}-\bar{q}) = 1 \implies \delta^{(3)}(\bar{p}-\bar{q}) \rightarrow \left(\frac{L}{2\pi}\right)^3 \delta_{\bar{p}_{\bar{n}}, \bar{q}_{\bar{m}}}$$

$$(2\pi)^3 \delta^{(3)}(\bar{p}=0) \rightarrow V$$

Fock space

vacuum $|0\rangle$, $a_p|0\rangle = 0$

a_p – annihilation operator , a^+ – creation operator

n-particle state : $$|\bar{p}_1, \bar{p}_2, \dots, \bar{p}_n\rangle \equiv \underbrace{(2E_1)^{1/2} \dots (2E_n)^{1/2}}_{\text{normalization}} a^+_{p_1} \dots a^+_{p_n} |0\rangle$$

$$|\bar{p}\rangle = (2E_p)^{1/2} a_p^+ |0\rangle$$

$$\langle \bar{p}_1, \bar{p}_2\rangle = (2E_1)^{1/2}(2E_2)^{1/2}\langle 0|\underbrace{a_1 a_2^+}_{(a_1 a_2^+ - a_2^+ a_1) + a_2^+ a_1}|0\rangle = (2E_1)^{1/2}(2E_2)^{1/2}\langle 0|\underbrace{[a_1, a_2^+]}_{(2\pi)^3\delta^{(3)}(\bar{p}_1 - \bar{p}_2)}|0\rangle = 2E_1 (2\pi)^3 \delta^{(3)}(\bar{p}_1 - \bar{p}_2)$$

– Find the Hamiltonian operator in terms of a_p and a_p^+

The energy of a field configuration $E[\phi] = \int d^3x \{ (\dot{\phi})^2 + (\bar{\nabla}\phi)^2 + V(\phi) \}$

$$\underbrace{\hspace{3cm}}_{T^{00}}$$

$$\hat{H} = \int d^3x \left\{ \frac{1}{2}(\dot{\hat{\phi}})^2 + \frac{1}{2}(\bar{\nabla}\hat{\phi})^2 + V(\hat{\phi}) \right\} \qquad \phi(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \left(a_p e^{-ipx} + a_p^+ e^{+ipx} \right)$$

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

$$\dot{\phi}(t,\bar{x}') = \pi(t,\bar{x}') = \int \frac{d^3p'}{(2\pi)^3 (2E_{p'})^{1/2}} (-iE_{p'}) \left(a_{p'} e^{-ip'x'} - a_{p'} e^{ip'x'} \right)$$

i) $$\int d^3x (\dot{\phi})^2 = \int d^3x \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \int \frac{d^3p'}{(2\pi)^3 (2\pi)^{1/2}} (-iE_p)(-iE_{p'}) \left(a_p e^{-ipx} - a_p^+ e^{ipx} \right)\left(a_{p'} e^{-ip'x} - a_{p'}^+ e^{ip'x} \right) =$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} (-iE_p) \int \frac{d^3p'}{\cancel{(2\pi)^3} (2E_{p'})^{1/2}} (-iE_{p'}) \cancel{(2\pi)^3} \Big\{ a_p a_{p'} \delta^{(3)}(-\bar{p}-\bar{p}') e^{-i(E_p+E_{p'})t} - a_p a_{p'}^+ \delta^{(3)}(\bar{p}'-\bar{p}) e^{-i(E_p-E_{p'})t} +$$

$$- a_p^+ a_{p'} \delta^{(3)}(\bar{p}-\bar{p}') e^{i(E_p-E_{p'})t} + a_p^+ a_{p'}^+ \delta^{(3)}(\bar{p}+\bar{p}') e^{i(E_p+E_{p'})t} \Big\} =$$

$$= \text{———} \;\; '' \;\; \text{———} \times \frac{1}{(2E_p)^{1/2}} (-iE_p) \left(a_p a_{-p} e^{-2iE_pt} - a_p a_p^+ - a_p^+ a_p + a_p^+ a_{-p}^+ e^{2iE_pt} \right) =$$

$$= \int \frac{d^3p}{(2\pi)^3} (-) \frac{E_p}{2} \left(a_p a_{-p} e^{-2iE_pt} - a_p a_p^+ - a_p^+ a_p + a_p^+ a_{-p}^+ e^{2iE_pt} \right)$$

ii) $$\int d^3x (\bar{\nabla}\phi)^2 =$$

$$\bar{\nabla}\phi = \bar{\nabla} \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \left(a_p e^{-i(E_pt - \bar{p}\bar{x})} + a_p^+ e^{i(E_pt - \bar{p}\bar{x})} \right) =$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} (i\bar{p}) \left(a_p e^{-ipx} - a_p^+ e^{+ipx} \right)$$

$$\int d^3x \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} (i\bar{p}) \int \frac{d^3p'}{(2\pi)^3 (2E_{p'})^{1/2}} (i\bar{p}') \left(a_p e^{-ipx} - a_p^+ e^{+ipx} \right)\left(a_{p'} e^{-ip'x} - a_{p'}^+ e^{ip'x} \right) =$$

$$\int d^3x \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} (i\bar{p}) \int \frac{d^3p'}{(2\pi)^3 (2E_{p'})^{1/2}} (i\bar{p}') \left(a_p e^{-ipx} - a_p^{+} e^{+ipx} \right) \left(a_{p'} e^{-ip'x} - a_{p'}^{+} e^{ip'x} \right) =$$

$$= \text{———} \;\text{"}\; \text{———} \;\text{"}\; \text{———} \;\text{"}\; \text{———} \Big(a_p a_{p'} e^{-i(p+p')x} - a_p a_{p'}^{+} e^{-i(p-p')x} +$$

$$- a_p^{+} a_{p'} e^{i(p-p')x} + a_p^{+} a_{p'}^{+} e^{i(p+p')x} \Big) =$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} (i\bar{p}) \int \frac{d^3p'}{\cancel{(2\pi)^3} (2E_{p'})^{1/2}} (i\bar{p}') \cancel{(2\pi)^3} \Big(a_p a_{p'} e^{-i(E_p+E_{p'})t} \delta^{(3)}(\bar{p}+\bar{p}') - a_p a_{p'}^{+} e^{-i(E_p-E_{p'})t} \delta^{(3)}(\bar{p}-\bar{p}') +$$

$$- a_p^{+} a_{p'} e^{i(E_p-E_{p'})t} \delta^{(3)}(\bar{p}-\bar{p}') + a_p^{+} a_{p'}^{+} e^{i(E_p+E_{p'})t} \delta^{(3)}(\bar{p}+\bar{p}') \Big) =$$

$$= \int \frac{d^3p}{(2\pi)^3} (-) \frac{\bar{p}^2}{2E_p} \left(-a_p a_{-p} e^{-2iE_pt} - a_p a_p^{+} - a_p^{+} a_p - a_p^{+} a_{-p}^{+} e^{2iE_pt} \right)$$

iii) $$\int d^3x \, \frac{1}{2} m^2 \phi^2 = \frac{m^2}{2} \int d^3x \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \int \frac{d^3p'}{(2\pi)^3 (2E_{p'})^{1/2}} \Big(a_p a_{p'} e^{-i(p+p')x} + a_p a_{p'}^{+} e^{-i(p-p')x} +$$

$$+ a_p^{+} a_{p'} e^{i(p-p')x} + a_p^{+} a_{p'}^{+} e^{i(p+p')x} \Big) =$$

$$= \frac{m^2}{2} \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \int \frac{d^3p'}{\cancel{(2\pi)^3} (2E_{p'})^{1/2}} \cancel{(2\pi)^3} \Big(a_p a_{p'} e^{-i(E_p+E_{p'})t} \delta^{(3)}(\bar{p}+\bar{p}') + a_p a_{p'}^{+} e^{-i(E_p-E_{p'})t} \delta^{(3)}(\bar{p}-\bar{p}') +$$

$$+ a_p^{+} a_{p'} e^{i(E_p-E_{p'})t} \delta^{(3)}(\bar{p}-\bar{p}') + a_p^{+} a_{p'}^{+} e^{i(E_p+E_{p'})t} \delta^{(3)}(\bar{p}+\bar{p}') \Big) =$$

$$= \frac{m^2}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(a_p a_{-p} e^{-2iE_p} + a_p a_p^{+} + a_p^{+} a_p + a_p^{+} a_{-p}^{+} e^{2iE_p} \right)$$

Total:

Total:

$$\int d^3x\,(\dot\phi)^2 = \int \frac{d^3p}{(2\pi)^3}\,(-)\,\frac{E_p}{2}\left(a_p a_{-p}\, e^{-2iE_pt} - a_p a_p^\dagger - a_p^\dagger a_p + a_p^\dagger a_{-p}^\dagger\, e^{2iE_pt}\right)$$

$$\int d^3x\,(\bar\nabla\phi)^2 = \int \frac{d^3p}{(2\pi)^3}\,(-)\,\frac{\bar p^2}{2E_p}\left(-a_p a_{-p}\, e^{-2iE_pt} - a_p a_p^\dagger - a_p^\dagger a_p - a_p^\dagger a_{-p}^\dagger\, e^{2iE_pt}\right)$$

$$H = -\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\Big\{ a_p a_{-p}\, e^{-2iE_pt}\left(E_p - \frac{\bar p^2}{E_p} - \frac{m^2}{E_p}\right)$$
$$+\, a_p a_p^\dagger\left(-E_p - \frac{\bar p^2}{E_p} - \frac{m^2}{E_p}\right)$$
$$+\, a_p^\dagger a_p\left(-E_p - \frac{\bar p^2}{E_p} - \frac{m^2}{E_p}\right)$$
$$+\, a_p^\dagger a_{-p}^\dagger\, e^{2iE_pt}\left(E_p - \frac{\bar p^2}{E_p} - \frac{m^2}{E_p}\right)\Big\} = +\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,E_p\,\overbrace{\left(a_p a_p^\dagger + a_p^\dagger a_p\right)}^{a_p a_p^\dagger - a_p^\dagger a_p + 2a_p^\dagger a_p} =$$

$$= \int \frac{d^3p}{(2\pi)^3}\,E_p\left(\frac{1}{2}\underbrace{[a_p, a_p^\dagger]}_{=(2\pi)^3\delta^{(3)}(0)} + a_p^\dagger a_p\right)$$

$$E_p + \frac{\bar p^2}{E_p} + \frac{m^2}{E_p} = \frac{1}{E_p}\left(E_p^2 + \bar p^2 + \underbrace{m^2}_{=E_p^2-\bar p^2}\right) = 2E_p$$

$$E_p - \frac{\bar p^2}{E_p} - \frac{m^2}{E_p} = \frac{1}{E_p}\left(\underbrace{E_p^2 - \bar p^2}_{m^2} - m^2\right) = 0$$

$$E_{vac} = \langle 0|H|0\rangle = \int\frac{d^3p}{(2\pi)^3}\,E_p\left(\frac{1}{2}\langle 0|[a_p, a_p^\dagger]|0\rangle\right) =$$

$$H = \int \frac{d^3p}{(2\pi)^3}\,E_p\; a_p^\dagger a_p$$

$$= \int\frac{d^3p}{(2\pi)^3}\,E_p\,\frac{1}{2}\,\delta^{(3)}(0)\,(2\pi)^3$$

in the finite volume $(2\pi)^3\delta^{(3)}(0) \to V$, so

$$E_{vac} = \frac{V}{2}\int\frac{d^3p}{(2\pi)^3}\,E_p \longrightarrow \infty$$

$$E_p = (m^2 + \bar p^2)^{1/2}$$

ultraviolate cutoff: $|\bar{p}| < \Lambda$, then

$$\rho_{vac} \equiv \frac{E_{vac}}{V} = \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} \sim \int_0^\Lambda dp\, p^3 \sim \Lambda^4$$

- Since we measure only energy differences we can subtract the constant vacuum contribution, then

$$H = \int \frac{d^3p}{(2\pi)^3} E_p\, a_p^+ a_p \quad \Rightarrow \quad \langle 0|H|0\rangle = 0$$

$$\int d^3p \to \left(\frac{2\pi}{L}\right)^3 \sum_{\bar{n}}$$

$$\int d^3p\, \delta^{(3)}(\bar{p}-\bar{q}) = 1 \quad \Rightarrow \quad \delta^{(3)}(\bar{p}-\bar{q}) \to \left(\frac{L}{2\pi}\right)^3 \delta_{\bar{p}_{\bar{n}}, \bar{q}_{\bar{m}}}$$

$$(2\pi)^3 \delta^{(3)}(\bar{p}=0) \to V$$

$$[a_p, a_q^+] = (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{q}) \longrightarrow L^3 \delta_{\bar{p},\bar{q}}$$

- Normal ordering $:\,0\,:$

$$:a_p a_p^+: = a_p^+ a_p \quad \text{and} \quad :a_p^+ a_p: = a_p^+ a_p$$

$$H = \frac{1}{2}\int d^3x : \left(\pi^2 + (\bar{\nabla}\phi)^2 + m^2\phi\right): = \int \frac{d^3p}{(2\pi)^3} E_p\, a_p^+ a_p$$

Number operator $N \equiv \int \frac{d^3p}{(2\pi)^3} a_p^+ a_p$

$$N|\bar{p}_1\rangle = \int \frac{d^3p}{(2\pi)^3} a_p^+ a_p \underbrace{(2E_{p_1})^{1/2} a_{p_1}^+ |0\rangle}_{|p_1\rangle} = (2E_{p_1})^{1/2} \int \frac{d^3p}{(2\pi)^3} a_p^+ \Big[\underbrace{(a_p a_{p_1}^+ - a_{p_1}^+ a_p)}_{[a_p, a_{p_1}^+] = (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{p}_1)} + a_{p_1}^+ a_p\Big]|0\rangle =$$

$$N|\bar{p}_1\rangle = 1 \cdot |\bar{p}_1\rangle \Leftarrow \begin{cases} = (2E_{p_1})^{1/2} \int \frac{d^3p}{\cancel{(2\pi)^3}} \cancel{(2\pi)^3} \delta^{(3)}(\bar{p}-\bar{p}_1)\, a_p^+|0\rangle = \\ = (2E_{p_1})^{1/2} a_{p_1}^+|0\rangle = |\bar{p}_1\rangle \end{cases}$$

Show that $N|p_1, p_2, \dots, p_n\rangle = n\,|p_1, p_2, \dots, p_n\rangle$

Show that $H|\bar{p}_1, \dots, \bar{p}_n\rangle = (E_{p_1} + \dots + E_{p_n})|\bar{p}_1, \dots, \bar{p}_n\rangle$

$$H|\bar{p}_1,\dots,\bar{p}_n\rangle = (2E_1)^{1/2}\cdots(2E_n)^{1/2}\int\frac{d^3p}{(2\pi)^3}\,E_p\,a_p^+\underbrace{a_p\,a_{p_1}^+}_{\underbrace{[a_p,a_{p_1}^+]}_{\overset{\shortparallel}{(2\pi)^3\delta^{(3)}(\bar{p}-\bar{p}_1)}}+a_{p_1}^+a_p}\cdots a_{p_n}^+|0\rangle$$

$$\left(E_{p_1}\,a_{p_1}^+a_{p_2}^+\cdots a_{p_n}^+|0\rangle + \int\frac{d^3p}{(2\pi)^3}E_p\,\underbrace{a_p^+a_{p_1}^+a_p}_{a_{p_1}^+a_p^+\left(\underbrace{[a_p,a_{p_2}^+]}_{(2\pi)^3\delta^{(3)}(\bar{p}-\bar{p}_2)}+a_{p_2}^+a_p\right)}\cdots a_{p_n}^+|0\rangle\right)$$

etc

Momentum operator

$$\hat{p}^i = \int d^3x :T^{0i}(\hat{\phi}): = \int d^3x :\partial_0\phi\,\partial^i\phi: = \int\frac{d^3p}{(2\pi)^3}\,p^i\,a_p^+a_p$$

Complex scalar field

$$\phi(x) = \int\frac{d^3p}{(2\pi)^3(2E_p)^{1/2}}\left(a_p\,e^{-ipx} + b_p^+\,e^{ipx}\right) \qquad p^0 = E_p = (m^2+\bar{p}^2)^{1/2}$$

$$\phi^+(x) = \text{---}\,''\,\text{---}\left(a_p^+\,e^{ipx} + b_p\,e^{-ipx}\right)$$

$$[\phi(t,\bar{x}),\pi(t,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y}) \implies [a_p,a_q^+] = [b_p,b_q^+] = (2\pi)^3\delta^{(3)}(\bar{p}-\bar{q})$$

$$[a_p,a_q] = [b_p,b_q] = [a_p^+,a_q^+] = [b_p^+,b_q^+] = 0$$

$$[a_p,b_q] = [a_p,b_q^+] = [a_p^+,b_q] = [a_p^+,b_p^+] = 0$$

vacuum: $a_p|0\rangle = b_p|0\rangle = 0$

– Find the energy-momentum tensor for complex scalar field (U(1) symmetry)

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 \qquad T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i$$

$$\phi_1 \equiv \phi\,,\quad \phi_2 \equiv \phi^*$$

$$T^{\mu\nu} = -\eta^{\mu\nu}\left(\partial_\alpha \phi^* \partial^\alpha \phi - m^2 \phi^* \phi\right) + \partial^\mu \phi^* \partial^\nu \phi \cdot 2$$

$$T^{00} = -\partial_\alpha \phi^* \partial^\alpha \phi + 2\partial^0\phi^*\partial^0\phi + m^2 \phi^*\phi = |\dot\phi|^2 + |\nabla\phi|^2 + m^2|\phi|^2$$

$$H = \int d^3x\, T^{00} = \int d^3x \left(|\dot\phi|^2 + |\nabla\phi|^2 + m^2\phi^*\phi\right)$$

$$\downarrow$$

$$\hat H = \int d^3x : \left(\dot\phi^\dagger \dot\phi + (\nabla\phi)^\dagger \nabla\phi + m^2 \phi^\dagger\phi\right) : \longrightarrow \int \frac{d^3p}{(2\pi)^3} E_p \left(a_p^\dagger a_p + b_p^\dagger b_p\right)$$

– U(1) conserved current

$$\left.\begin{aligned} j^\mu &= i\phi^* \overleftrightarrow{\partial^\mu}\phi = i\phi^*\partial^\mu\phi - i(\partial^\mu\phi^*)\phi \\ Q_{U(1)} &= \int d^3x\, j^0 = i\int d^3x\, \phi^* \overleftrightarrow{\partial^0}\phi \end{aligned}\right] \text{classical}$$

U(1) charge operator $\hat Q_{U(1)} = i\int d^3x : \phi^\dagger \overleftrightarrow{\partial^0} \phi :$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right) \qquad \phi^\dagger(x) = \int \frac{d^3p'}{(2\pi)^3(2E_{p'})^{1/2}}\left(a_{p'}^\dagger e^{ip'x} + b_{p'} e^{-ip'x}\right)$$

$$\hat Q_{U(1)} = i\int d^3x \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \int \frac{d^3p'}{(2\pi)^3(2E_{p'})^{1/2}} : \Big[\left(a_{p'}^\dagger e^{ip'x} + b_{p'} e^{-ip'x}\right)(-iE_p)\left(a_p e^{-ipx} - b_p^\dagger e^{ipx}\right) +$$

$$- iE_{p'}\left(a_{p'}^\dagger e^{ip'x} - b_{p'} e^{-ip'x}\right)\left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right)\Big] : =$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \frac{d^3p'}{\cancel{(2\pi)^3} (2E_{p'})^{1/2}} \cancel{(2\pi)^3} : \Big[a_{p'}^\dagger a_p (E_p + E_{p'})\,\delta^{(3)}(\bar p' - \bar p) + a_{p'}^\dagger b_p^\dagger \underbrace{(-E_p + E_{p'})\, e^{2iE_p t}\,\delta^{(3)}(\bar p' + \bar p)}_{\longrightarrow 0} +$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \frac{d^3p'}{\cancel{(2\pi)^3} (2E_{p'})^{1/2}} \cancel{(2\pi)^3} : \Big[a^+_{p'} a_p (E_p + E_{p'}) \delta^{(3)}(\vec{p}' - \vec{p}) + a^+_{p'} b^+_p \underbrace{(-E_p + E_{p'}) e^{2iE_p t} \delta^{(3)}(\vec{p}' + \vec{p})}_{\longrightarrow 0} +$$

$$+ b_{p'} a_p \underbrace{(E_p - E_{p'}) e^{-2iE_{p'} t} \delta^{(3)}(\vec{p}' + \vec{p})}_{\longrightarrow 0} - b_{p'} b^+_p (E_p + E_{p'}) \delta^{(3)}(\vec{p}' - \vec{p}) \Big] :$$

$$= \int \frac{d^3p}{(2\pi)^3 \cancel{(2E_p)^{1/2}}} \frac{1}{\cancel{(2E_p)^{1/2}}} \cancel{2E_p} : (a^+_p a_p - b_p b^+_p) : = \int \frac{d^3p}{(2\pi)^3} (\underbrace{a^+_p a_p}_{} - \underbrace{b^+_p b_p}_{})$$

antiparticles

number operator of particles created by a^+

- Show that

$$Q_{U(1)} \, a^+_p |0\rangle = + a^+_p |0\rangle$$

$$Q_{U(1)} \, b^+_p |0\rangle = - b^+_p |0\rangle$$

Propagators

Time-ordered product: $T\{\phi(y)\phi(x)\} = \begin{cases} \phi(y)\phi(x) & \text{for } y^0 > x^0 \\ \phi(x)\phi(y) & \text{for } y^0 < x^0 \end{cases}$

In other words

$$T\{\phi(y)\phi(x)\} = \phi(y)\phi(x)\,\Theta(y^0 - x^0) + \phi(x)\phi(y)\,\Theta(x^0 - y^0) \quad \text{for} \quad \Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

The Feynman propagator $D(x-y) \equiv \langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle$

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} a_p e^{-ipx} \quad , \quad \phi^-(x) \equiv \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} a^+_p e^{ipx} \qquad \left(\phi^-(x) = (\phi^+(x))^+ \right)$$

$$\phi^+(x)|0\rangle = 0 \quad \text{and} \quad \langle 0 | \phi^-(x) = 0$$

1) $x^0 > y^0$

$$T\{\phi(x)\phi(y)\} = \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) =$$

$$= \phi^+(y)\,\phi^+(y) + \phi^-(y)\,\phi^+(x) + \phi^-(x)\,\phi^+(y) + \phi^-(x)\,\phi^-(y) + [\phi^+(x), \phi^-(y)] =$$

$$= :\phi(x)\,\phi(y): + [\phi^+(x), \phi^-(y)]$$

2) $x^0 < y^0$

$$T\{\phi(x)\,\phi(y)\} = \phi^+(y)\,\phi^+(x) + \phi^+(y)\,\phi^-(x) + \phi^-(y)\,\phi^+(x) + \phi^-(y)\,\phi^-(x) =$$

$$= \phi^+(x)\,\phi^+(y) + \phi^-(x)\,\phi^+(y) + \phi^-(y)\,\phi^+(x) + \phi^-(x)\,\phi^-(y) + [\phi^+(y), \phi^-(x)] =$$

$$= :\phi(x)\,\phi(y): + [\phi^+(y), \phi^-(x)]$$

$$\left.\begin{aligned} :\phi(x)\,\phi(y): &= \phi^+(x)\,\phi^+(y) + \phi^-(y)\,\phi^+(x) + \phi^-(x)\,\phi^+(y) + \phi^-(x)\,\phi^-(y) \\ :\phi(y)\,\phi(x): &= \phi^+(y)\,\phi^+(x) + \phi^-(x)\,\phi^+(y) + \phi^-(y)\,\phi^+(x) + \phi^-(y)\,\phi^-(x) \end{aligned}\right| \Rightarrow\ :\phi(x)\,\phi(y): = :\phi(y)\,\phi(x):$$

$$\Downarrow$$

$$T\{\phi(x)\,\phi(y)\} = :\phi(x)\,\phi(y): + D(x-y)\,,$$

where

$$D(x-y) = [\phi^+(y), \phi^-(x)]\,\Theta(y^0 - x^0) + \underbrace{[\phi^+(x), \phi^-(y)]}_{\text{c-number}}\,\Theta(x^0 - y^0)$$

Note that

$$\langle 0|\, :\phi(x)\,\phi(y):\, |0\rangle = 0$$

$$\langle 0|\, D(x-y)\, |0\rangle = D(x-y)\,\langle 0|0\rangle = D(x-y)$$

therefore indeed $D(x-y)$ is just the Feynman propagator

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \frac{d^3p'}{(2\pi)^3 (2E_{p'})^{1/2}}\, e^{-ipx + ip'y} \underbrace{[a_p, a^+_{p'}]}_{(2\pi)^3 \delta^{(3)}(\bar{p} - \bar{p}')} = \int \frac{d^3p}{(2\pi)^3\, 2E_p}\, e^{-i\overbrace{[E_p(x^0 - y^0) - \bar{p}(\bar{x} - \bar{y})]}^{p\cdot(x-y)}}$$

$$\Downarrow$$

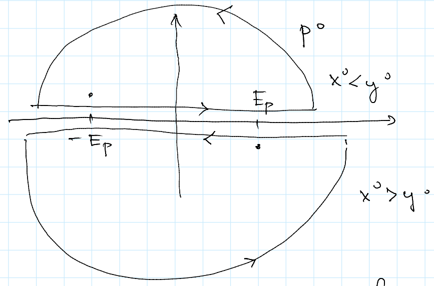
$$D(x-y) = \int \frac{d^3p}{(2\pi)^3\, 2E_p} \left\{ e^{-ip(x-y)}\,\Theta(x^0 - y^0) + e^{ip(x-y)}\,\Theta(y^0 - x^0) \right\}, \quad D(x-y) = \langle 0|\, T\{\phi(x)\,\phi(y)\}\, |0\rangle$$

– Show that $D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}\, e^{-ip(x-y)}$, where $\epsilon \to 0^+$

$$\int d^3p\; e^{i\bar{p}(\bar{x}-\bar{y})} \int^{+\infty} dp^0 \quad i \quad e^{-ip^0(x^0 - y^0)} \quad \text{where} \quad E = (m^2 + \bar{p}^2)^{1/2}$$

$$\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}(\vec{x}-\vec{y})} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi}\, \frac{i}{(p^0)^2-E_p^2+i\epsilon}\, e^{-ip^0(x^0-y^0)} \qquad \text{where } E_p = (m^2+\vec{p}^{\,2})^{1/2}$$

poles: $p^0 = \pm\left(E_p^2 - i\epsilon\right)^{1/2} = \pm E_p\left(1 - \frac{i\epsilon}{2E_p^2} + \dots\right)$



$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, a_k)$$

1) $x^0<y^0$

$$\frac{i}{2\pi}\, 2\pi i\, \frac{-1}{2E_p}\, e^{+iE_p(x^0-y^0)} = \frac{e^{iE_p(x^0-y^0)}}{2E_p} \qquad \frac{1}{p^{0\,2}-E_p^2+i\epsilon} = \left(\frac{-1}{p^0+E_p-i\epsilon} + \frac{+1}{p^0-E_p+i\epsilon}\right)\frac{1}{2E_p}$$

2) $x^0>y^0$

$$\frac{i}{2\pi}\, 2\pi i\,(-)\, \frac{1}{2E_p}\, e^{-iE_p(x^0-y^0)} = \frac{e^{-iE_p(x^0-y^0)}}{2E_p}$$

□

The Feynman propagator in the momentum space

$$\tilde{D}(p) = \frac{i}{p^2-m^2+i\epsilon}$$

– Show that $D(x-y)$ is a Green's function of $(\Box_x+m^2)$, so that

$$(\Box_x+m^2)\,D(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m^2+i\epsilon}\left[(-ip)^2+m^2\right] e^{-ip(x-y)} =$$

$$= -i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} = -i\,\delta^{(4)}(x-y)$$